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The continuous spectrum of the Orr–Sommerfeld equation. Part 1. The spectrum and the eigenfunctions

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It is shown that the Orr–Sommerfeld equation, which governs the stability of any mean shear flow in an unbounded domain which approaches a constant velocity in the far field, has a continuous spectrum. This result applies to both the temporal and the spatial stability problem. Formulae for the location of this continuum in the complex wave-speed plane are given. The temporal continuum eigenfunctions are calculated for two sample problems: the Blasius boundary layer and the two-dimensional laminar jet. The nature of the eigenfunctions, which are very different from the Tollmien–Schlichting waves, is discussed. Three mechanisms are proposed by which these continuum modes could cause transition in a shear flow while bypassing the usual linear Tollmien–Schlichting stage.

1. Introduction

The problem of the stability of laminar flow against externally imposed disturbances and the connexion between instability, if it exists, and the transition from laminar to turbulent flow has been studied for nearly a century and remains one of the central problems in fluid dynamics. Ever since Reynolds' classic experiments (1883) it has been conjectured that transition from laminar to turbulent flow is the result of an instability in the laminar flow. Beginning with Rayleigh (1880), there have been many theoretical studies which have attempted to predict under what conditions small disturbances in the velocity profile would grow or decay.

The main interest in stability calculations centres on the stability of boundary-layer flows and related flows such as jets, wakes and free shear flows; Couette and particularly Poiseuille flow have had the status of 'canonical' problems. Tollmien (1931), Schlichting (1951) and Lin (1955) made notable advances in the analytical calculation of the stability of these parallel shear flows to two-dimensional wavelike disturbances, the Tollmien–Schlichting waves (hereafter called TS waves). The restriction to two-dimensional waves was justified by a result of Squire (1933), who showed that, if an oblique wave is unstable at some Reynolds number, then a two-dimensional TS wave is unstable at a smaller Reynolds number. Thus, in order to find the critical Reynolds number it suffices to consider only two-dimensional TS waves.

Although the theory was highly developed, it was, prior to the early 1940's, generally regarded as irrelevant to the actual mechanics of transition because TS waves had never been observed experimentally, and it appeared that any disturbance, whether in the free stream, in the boundary layer or on the boundary, could cause a transition if it was large enough or if the Reynolds number was high enough.

The experimental studies of Schubauer & Skramstad (1941, 1947, 1948) first showed that, if the free-stream disturbances, wall vibrations etc. were sufficiently small, the appearance and growth of TS waves could play a dominant role in boundary-layer transition. They found generally good agreement between the predictions of linear stability theory and the experimental results, not only for the shape of the neutral-stability curve and the phase speeds of the TS waves, but also for the shape of the TS waves as a function of distance from the boundary.

While the work of Schubauer & Skramstad showed the essential correctness of linear stability theory, the calculations of the detailed stability characteristics for a particular profile remained quite delicate in that the results obtained were sensitive to the detailed shape of the profiles and the approximations used in the asymptotic analysis. It is only the development of sophisticated numerical methods and the high-speed computers to implement them (Brown 1959; Kurtz & Crandall 1962; Nachtsheim 1963; Kaplan 1964; Gallagher & Mercer 1964; Mack 1965; Landahl 1966; Radbill & Van Driest 1966; Wazzan, Okamura & Smith 1968; Lessen, Sadler & Liu 1968; Grosch & Salwen 1968; Orszag 1971; Salwen & Grosch 1972) that has made these calculations relatively routine.

It is usually assumed that the eigenvalue problem of linear stability theory has an infinite set of discrete eigenvalues and a corresponding infinite set of eigenfunctions. Assuming the completeness of this set of eigenfunctions, an arbitrary initial disturbance which satisfies the boundary conditions can be expanded in terms of them. We know of no general proof of this completeness 'theorem'.

Almost all stability calculations have been concerned with finding the first or least stable eigenvalue. It should be noted that, if expansion (spectral) techniques (Grosch & Salwen 1968; Orszag 1971; Salwen & Grosch 1972) are used to solve the eigenvalue problem in a finite domain, a number of the higher eigenvalues are found numerically along with the first. If a sufficient number of terms is used in the expansion, these numerically calculated eigenvalues will overlap the asymptotic (high-order) distribution of eigenvalues, for which analytic formulae may be derived, and the complete spectrum may be found. Apart from any difficulties in the analysis or calculation, this concentration on the first eigenvalue may be due to the facts that (i) experiments suggest that there is only a single unstable mode and (ii) for those flows for which the higher modes have been calculated, i.e. plane Couette (Gallagher & Mercer 1964), plane Poiseuille (Grosch & Salwen 1968; Orszag 1971) and pipe Poiseuille flow (Lessen *et al.* 1968; Salwen & Grosch 1972), these higher modes have been found to be highly damped.

Very recently Jordinson (1971), Mack (1976) and Corner, Houston & Ross (1976), using different numerical methods, have calculated the higher eigenvalues of the Orr–Sommerfeld equation for Blasius flow. Jordinson calculated eigenvalues for the spatially and temporally growing or decaying modes for a single Reynolds number and a single wavenumber (temporal problem) or a single frequency (spatial problem). Mack calculated the eigenvalues for a number of different values of the wavenumber and Reynolds number for the temporal stability problem. Corner *et al.* recalculated the spatial modes. While there is some disagreement in the number and location of the eigenvalues found, all of these authors agree that at any Reynolds number there is only a finite, and small, number of discrete eigenvalues. A finite set of eigenfunctions cannot be a complete set. How, then, can an arbitrary initial disturbance be expanded? These authors suggest that there is a continuous spectrum.

We know of no general results concerning the spectrum of the Orr–Sommerfeld equation for an arbitrary parallel shear flow in an unbounded domain. For the Orr–Sommerfeld equation in a bounded domain or in the case of the inviscid stability problem, for which the governing equation is the Rayleigh equation, a number of specific problems have been studied in detail, and some general results concerning the spectrum are available.

Much is known about the spectrum of the inviscid stability problem (the Rayleigh problem). Howard's circle theorem (1961) states that all eigenvalues of the Rayleigh problem lie in a circle in the complex plane whose diameter is the range of the undisturbed velocity. Case (1960) has studied the spectrum of inviscid plane Couette flow and shown that there are no discrete eigenvalues; there is only a continuum. In the same paper Case generalized this result and showed that, for any mean velocity profile in a finite domain, the Rayleigh problem always has a continuous spectrum in addition to the discrete spectrum, if one exists. Rosencrans & Sattinger (1966) and Sattinger (1967) have proved, under fairly general conditions on the velocity profile of the mean flow, that the Rayleigh problem in a finite domain has only a finite number of discrete eigenvalues. Plane Couette and Poiseuille flows and, it appears, most other physically reasonable flows in a finite domain satisfy the conditions of this theorem.

Case (1961) and Lin (1961) have examined the connexion between the spectrum of the inviscid problem and that of the viscous problem for a finite domain. Case has shown that, if the initial conditions are independent of the viscosity (Reynolds number), the solution of the viscous problem approaches the solution of the inviscid problem as the viscosity goes to zero and hence has a continuous spectrum. Lin has pointed out that, for the viscous problem, the initial conditions in general depend on the viscosity and, therefore, Case's theorem does not apply. Lin has further shown that the Orr–Sommerfeld equation has only discrete eigenvalues in a bounded domain.

It has been proved that the Orr–Sommerfeld equation has a complete set of eigenfunctions for the specific cases of plane Couette flow (Haupt 1912) and plane Poiseuille flow (Schensted 1960). DiPrima & Habetler (1969) have proved a completeness theorem for a class of non-self-adjoint eigenvalue problems in a bounded domain. Using this theorem they showed that the Bénard problem (linear stability of a layer of fluid heated from below), the Taylor problem (linear stability of the flow between rotating concentric cylinders) and the Orr–Sommerfeld equation for any flow in a bounded domain have a complete set of eigenfunctions.

None of these results apply to the Orr–Sommerfeld equation if the domain is infinite as it is for Blasius flow. The possibilities for the spectrum of the Orr–Sommerfeld equation in an infinite domain are (a) that there is an infinite set of discrete eigenvalues without a continuum, (b) that there is an infinite set of discrete eigenvalues with a continuum, (c) that there is a finite number of eigenvalues without a continuum or (d) that there is a finite number of discrete eigenvalues with a continuum.

In this paper we shall *not* be concerned with whether or not the discrete spectrum of the Orr–Sommerfeld equation is finite in an infinite domain. We shall consider the existence of a continuous spectrum of the Orr–Sommerfeld equation for both the temporal and the spatial problem. In § 2, it is shown that the Orr–Sommerfeld equation has a continuous spectrum for a wide class of shear flows in an infinite domain. Formulae for the continuous spectrum in terms of the wavenumber or frequency and the Reynolds number are given for these flows. Detailed calculations of the continuous spectrum and examples of the continuum eigenfunctions for two specific flows, the Blasius boundary layer and the two-dimensional laminar jet, are given in § 3. Also included in § 3 is a physical interpretation of the modes. Section 4 contains some speculation on mechanisms by which these modes could lead to transition. Finally, § 5 is a brief summary of our results.

In a future paper (part 2) we shall consider detailed representation of particular disturbances in terms of these eigenfunctions and their spatial and temporal evolution. We also plan to study at some later time the mechanisms (suggested in § 4) by which these modes might lead to transition.

2. General analysis

2.1. *Formulation*

The basic flow is a parallel shear flow $(U(y),\ 0,\ 0)$ in a Cartesian co-ordinate system $(x,\ y,\ z)$. We consider infinitesimal two-dimensional disturbances to this flow. The stream function of the disturbance is assumed to be of the form

$$\psi(x,y,t) = \phi(y)\, e^{i\alpha(x-ct)}. \tag{1}$$

As is well known, with these assumptions the linearized Navier–Stokes equations reduce to the Orr–Sommerfeld equation:

$$\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\phi = i\alpha R\left[(U-c)\left(\frac{d^2}{dy^2}-\alpha^2\right)-\left(\frac{d^2U}{dy^2}\right)\right]\phi. \tag{2}$$

All variables are dimensionless; the length scale is L, the velocity scale is U_0 and the time scale L/U_0. As usual, α is the wavenumber, c the phase speed and R the Reynolds number U_0L/ν, with ν the kinematic viscosity. Equation (2) is to be solved with suitable boundary conditions (discussed below). In the temporal stability problem α is real and c is complex, so that the flow is unstable if $\operatorname{Im} c > 0$. For the spatial stability problem,

$$\omega = \alpha c \tag{3}$$

is real and α is complex; the flow is unstable if $\operatorname{Im} \alpha < 0$.

The disturbance (1) is just a single eigenfunction of the stability problem. If, as is the case for the temporal stability problem of plane Poiseuille flow, the Orr–Sommerfeld equation has an infinite set of discrete eigenvalues $\{c_n\}$ and a corresponding complete set of eigenfunctions $\{\phi_n(y)\}$, the most general solution of the linearized Navier–Stokes equations has, at fixed R, a stream function of the form

$$\begin{aligned}\Psi(x,y,t) &= \int_{-\infty}^{\infty}\sum_{n=1}^{\infty} A_n(\alpha)\,\psi_n(x,y,t;\,\alpha)\,d\alpha \\ &= \int_{-\infty}^{\infty}\sum_{n=1}^{\infty} A_n(\alpha)\,\phi_n(y;\,\alpha)\exp\left[i\alpha(x-c_n(\alpha)\,t)\right]d\alpha,\end{aligned} \tag{4}$$

where the $A_n(\alpha)$ are 'Fourier' amplitudes. A physical disturbance in the flow will, in general, excite a mixture of modes, and this, for the case of temporal stability, corresponds to specifying $\Psi(x, y, 0)$. If any mode $\psi_n(x, y, t; \alpha)$ grows in time, then the initial disturbance, or rather part of it, grows in time.

In order to generate only a single mode with a given wavenumber α_0, $\Psi(x, y, 0)$ must be spatially periodic with wavenumber α_0.

For an aperiodic initial disturbance the integral over α in (4) must be retained and the form of the disturbance at large t will be

$$\Psi \approx \int_{-\infty}^{\infty} A_1(\alpha)\,\phi_1(y;\,\alpha)\exp i\alpha(x - c_1(\alpha))\,t\,d\alpha, \tag{5}$$

where $\phi_1(y;\,\alpha)$ is the mode with the maximum $\alpha \operatorname{Im} c$ and the major contribution will come from a band of wavenumbers near the fastest-growing mode.

Actually the temporal stability problem does not represent the controlled experiments that are carried out in shear flows. These experiments involve the generation of a periodic disturbance of fixed frequency at some fixed x position, say x_0, and observation of the spatial growth or decay of the disturbance, in this case

$$\Psi(x, y, t) = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} A_n(\omega)\,\phi_n(y;\,\omega)\exp[i(\alpha_n x - \omega t)]\,d\omega. \tag{6}$$

In principle $\Psi(x_0, y, t)$, which is periodic in time and is of the form

$$\Psi(x_0, y, t) = F(y)\exp(i\omega_0 t), \tag{7}$$

is known and so the $A_n(\omega)$ can be found. If, as appears to be the case for Blasius flow, the higher TS modes are highly damped compared with the fundamental TS mode, they decay rapidly with increasing x, and some distance downstream from the disturbance generator only the fundamental mode is observed.

Clearly the TS modes are only a mathematical device for representing physical disturbances. It is only an accident that the higher TS modes are highly damped, so that some distance downstream from a disturbance generator the pure, fundamental TS wave can be observed, even though a pure, fundamental TS wave is not generated by the physical disturbance generator. If the Orr–Sommerfeld equation has eigenfunctions other than the TS modes, they too must be regarded as mathematical representations of portions of physically realizable disturbances.

For a general shear flow, it is conceivable that the Orr–Sommerfeld equation might not have an infinite set of discrete eigenvalues and so could not possibly have a complete set of corresponding eigenfunctions or that, even if it had an infinite set of discrete eigenvalues, the corresponding infinite set of eigenfunctions might not be complete. In either of these cases completeness could require the inclusion of 'improper eigenfunctions' (Friedman 1956, p. 233) corresponding to a continuous spectrum. If this were the case, then the stream function could not, in general, be represented as a sum over only the discrete modes, as in (4); the contribution from the continuum must be included, i.e. for the temporal problem

$$\Psi = \int_{-\infty}^{\infty}\left\{\sum_n A_n(\alpha)\,\phi_n(y;\,\alpha)\exp[i\alpha(x - c_n t)] + \int B(\alpha, c)\,\phi(y;\,\alpha, c)\exp[i\alpha(x - ct)]\,dc\right\}d\alpha, \tag{8}$$

where the integral in the second term is taken over the continuum of c.

This latter case is quite common in other branches of physics. Consider, for example, the non-relativistic Schrödinger equation (Landau & Lifshitz 1959, p. 48) for an electron in a potential. If the potential is that of hydrogen, it is easily shown that there is an infinite set of discrete eigenvalues and corresponding eigenfunctions which are not complete. Completeness requires the inclusion of the continuous spectrum and 'improper' eigenfunctions. If the potential is that of the deuteron, there is only one discrete eigenvalue, and completeness requires the inclusion of the continuum modes (Bethe 1947, p. 34). In both these cases the discrete eigenvalues are the energy levels of the bound states of the electron, and a point on the continuum is the energy of an electron which is scattered by the potential.

A trivial example may serve to illustrate the essential point of this argument. Consider the wave equation

$$\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2. \tag{9}$$

If we look for solutions of the form

$$u(x,t) = f(x)\, e^{i\omega t}, \tag{10}$$

then $f(x)$ is a solution of

$$d^2 f/dx^2 + \omega^2 f = 0. \tag{11}$$

If the boundary conditions for (9) are

$$u(0,t) = u(1,t) = 0, \tag{12}$$

then there is an infinite set of discrete eigenvalues $\{\omega_n\}$ and eigenfunctions $\{f_n(x)\}$,

$$\omega_n = n\pi, \quad f_n(x) = 2^{-\frac{1}{2}} \sin(n\pi x), \quad n = 1, 2, 3, \ldots, \qquad (13), (14)$$

and these eigenfunctions are orthonormal and form a complete set. If we consider the infinite domain $0 \leqslant x < \infty$, instead of the finite domain $0 \leqslant x \leqslant 1$, and impose the boundary conditions

$$u(0,t) = 0, \tag{15a}$$

$$u(x,t) \to 0 \quad \text{as} \quad x \to \infty, \tag{15b}$$

it is obvious that there are a finite number of discrete eigenvalues; in fact, there are no eigenvalues. If, however, the boundary condition (15*b*) is relaxed to

$$u(x,t) \quad \text{bounded as} \quad x \to \infty, \tag{15c}$$

then the spectrum is a continuum, with ω real and $\omega \geqslant 0$, and the 'improper' eigenfunctions are

$$f(x;\, \omega) = (2\pi)^{-\frac{1}{2}} \sin \omega x. \tag{16}$$

These eigenfunctions are improper in the sense that they are not square integrable in $[0, \infty)$, but, as written, they are δ-function normalized, i.e.

$$\int_0^\infty f(x;\, \omega) f(x;\, \omega')\, dx = \delta(\omega - \omega'). \tag{17}$$

This example is, of course, trivial. The Fourier sine series on $[0, 1]$ has been replaced by the Fourier sine integral on $[0, \infty)$. The fact that the continuum eigenfunctions are improper (not square integrable in $[0, \infty)$) is so familiar as to be accepted without comment. The fact that $f(x;\, \omega)$ is not a physically realizable mode but only a mathematical representation of a portion of a wave packet, which is physically realizable, is a familiar concept.

It will be shown below that the Orr–Sommerfeld equation for a wide class of shear flows in an infinite domain always has a continuous spectrum and that the corresponding continuum or 'improper' eigenfunctions are δ-function normalizable. The first step in showing this requires an examination of the boundary conditions.

2.2. *Boundary conditions for the Orr–Sommerfeld equation*

The Orr–Sommerfeld equation (2) is to be solved with suitable boundary conditions in $[0, \infty)$. First consider the boundary conditions at $y = 0$.

If the flow is a boundary layer, i.e. there is a solid wall at $y = 0$, then the velocity components $(u, v, 0)$ of the disturbance must vanish at $y = 0$. Since

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x \tag{18}$$

two of the boundary conditions for a boundary layer are then

$$\phi = d\phi/dy = 0 \quad \text{at} \quad y = 0. \tag{19}$$

If the flow is an unbounded shear flow such as a jet, wake or free shear layer, ϕ can be written as a sum of symmetric and antisymmetric modes (symmetry about $y = 0$). If $U(y)$ is symmetric about $y = 0$, as is usual for a jet or wake, the symmetric and antisymmetric modes are uncoupled solutions of (2). However, if $U(y)$ is antisymmetric, as in a free shear layer, for example, the symmetric and antisymmetric modes are coupled. For ϕ symmetric about $y = 0$, the boundary conditions are

$$d\phi/dy = d^3\phi/dy^3 = 0 \quad \text{at} \quad y = 0, \tag{20}$$

while the boundary conditions for an antisymmetric mode are

$$\phi = d^2\phi/dy^2 = 0 \quad \text{at} \quad y = 0. \tag{21}$$

What boundary conditions are to be applied at infinity? It is almost universal to require that u and v vanish at infinity and hence that

$$\phi, d\phi/dy \to 0 \quad \text{as} \quad y \to \infty. \tag{22}$$

This condition ensures, of course, that

$$\int_0^\infty \phi^*\phi\, dy < \infty, \tag{23}$$

the star denoting the complex conjugate, i.e. it ensures that ϕ is in L_2. This boundary condition will not be used in this paper. Instead the weaker condition

$$\phi, d\phi/dy \quad \text{bounded as} \quad y \to \infty \tag{24}$$

will be imposed.

Clearly if u, $v \to 0$ as $y \to \infty$ they are also bounded as $y \to \infty$. But if there are modes that are bounded as $y \to \infty$ which do not satisfy (22) ($u, v \to 0$ as $y \to \infty$), then they are 'improper' eigenfunctions and must be regarded as a mathematical device for representing a class of physical disturbances. It will be shown below that the eigenvalues of the modes which satisfy (24) but not (22) form a continuum, and the sum over modes in (4) and (6) must be replaced by a sum over the discrete modes and an integral over the continuum as in (8).

2.3. *The continuous spectrum*

The temporal stability problem. Equation (2) is a fourth-order linear differential equation and therefore has four linearly independent solutions $\phi_j(y), j = 1, 2, 3, 4$. The asymptotic (i.e. $y \to \infty$) form of these solutions can be found by considering (2) as $y \to \infty$. For the class of shear flows we are considering,

$$U(y) \to U_1 \quad \text{(a constant)}, \quad U'(y), U''(y) \to 0 \tag{25}$$

as $y \to \infty$. If $U(y)$ does not approach a constant as $y \to \infty$, then there is no Galilean transformation under which the flow has finite energy. The constant U_1 is unity for a boundary layer, a wake or a shear layer and is zero for a jet.

Then, as $y \to \infty$, (2) becomes

$$\left(\frac{d^2}{dy^2}-\alpha^2\right)^2 \phi = i\alpha R\left[(U_1-c)\left(\frac{d^2}{dy^2}-\alpha^2\right)\right]\phi, \tag{26}$$

a fourth-order differential equation with constant coefficients. The four independent solutions of the Orr–Sommerfeld equation are then asymptotic to the solutions of (26), i.e.

$$\phi_j(y) \sim \exp(\lambda_j y), \tag{27}$$

$$\lambda_1 = -Q^{\frac{1}{2}}, \quad \lambda_2 = +Q^{\frac{1}{2}}, \quad \lambda_3 = -\alpha, \quad \lambda_4 = +\alpha, \tag{28a–d}$$

$$Q = i\alpha R(U_1-c)+\alpha^2. \tag{29}$$

ϕ_1 and ϕ_2 are the 'viscous' solutions while ϕ_3 and ϕ_4 are 'inviscid' solutions. For the temporal stability problem and a general complex c, λ_1 has a negative real part and λ_2 has a positive real part, so only ϕ_1 and ϕ_3 satisfy the boundary conditions (24). A linear combination of ϕ_1 and ϕ_3, the decaying viscous and inviscid solutions, also satisfies the more stringent boundary conditions (22).

The Orr–Sommerfeld equation (2) can then be solved for ϕ_1 and ϕ_3. The eigenvalue problem then reduces to finding a value of c for which a linear combination of ϕ_1 and ϕ_3 will satisfy the two boundary conditions at $y = 0$, i.e. (19), (20) or (21).

Note that all these solutions satisfy (22) as well as (24), i.e. they are in L_2. However there is another class of solutions, which satisfy (24) but not (22). To see the asymptotic ($y \to \infty$) form of these solutions, assume that Q is real and negative, i.e.

$$\operatorname{Re} Q = \alpha R c_i + \alpha^2 < 0, \tag{30}$$

$$\operatorname{Im} Q = \alpha R(U_1 - c_r) = 0. \tag{31}$$

From (30),

$$c_i < -\alpha/R, \tag{32}$$

which is written, for convenience, as

$$c_i = -(1+k^2)\alpha/R, \quad k \text{ real and non-zero.} \tag{33}$$

Since $\alpha R \neq 0$, (31) gives

$$c_r = U_1. \tag{34}$$

The trivial solution $\alpha = 0$ implies $\phi = 0$.

Now

$$c = U_1 - i(1+k^2)\alpha/R \tag{35}$$

gives

$$Q = -k^2\alpha^2, \tag{36}$$

$$\lambda_1 = -ik\alpha, \quad \lambda_2 = +ik\alpha. \tag{37a, b}$$

Therefore both the viscous solutions ϕ_1 and ϕ_2, as well as ϕ_3, the decaying inviscid solution, satisfy the boundary condition (24) as $y \to \infty$, and

$$\begin{aligned} \phi &= A\phi_1 + B\phi_2 + C\phi_3 \\ &\sim Ae^{-ik\alpha y} + Be^{ik\alpha y} + Ce^{-\alpha y} \quad \text{as} \quad y \to \infty. \end{aligned} \tag{38}$$

The independent solutions ϕ_1, ϕ_2 and ϕ_3 to (2) are found, and the boundary conditions (19), (20) or (21) at $y = 0$ are applied. Because there are three arbitrary constants and only two conditions to be satisfied, the boundary condition at $y = 0$ can always be satisfied.

The temporal stability problem for the Orr–Sommerfeld equation has, therefore, a continuous spectrum along the line

$$c = U_1 - i(1+k^2)\,\alpha/R \tag{39}$$

in the complex c plane for arbitrary real positive k. Now, $U_1 = 1$ for a (suitably normalized) boundary layer, wake and shear layer and $U_1 = 0$ for a jet. All of these continuum modes are thus damped ($c_i < 0$); however, the damping rate

$$\alpha c_i = -(1+k^2)\,\alpha^2/R \tag{40}$$

is quite small, at least for small k, because $\alpha^2/R \ll 1$ in most situations. The continuum eigenfunctions are of the form of (38), where two of the three constants are determined from the boundary conditions at $y = 0$ and the third is arbitrary.

When $k = 0$,

$$\lambda_1 = \lambda_2 = 0 \tag{41}$$

and

$$\phi_1 \sim 1, \quad \phi_2 \sim y, \tag{42a, b}$$

and ϕ is once again a linear combination of ϕ_1 and ϕ_3. In general this value of c will not be an eigenvalue.

The spatial stability problem. The case of spatial stability can be treated in an exactly similar way. In this case ω is real, α is complex and we require that

$$\operatorname{Re} Q = \alpha_r^2 - \alpha_i^2 - RU_1\alpha_i < 0 \tag{43}$$

and

$$\operatorname{Im} Q = 2\alpha_r\alpha_i + RU_1\alpha_r - R\omega = 0. \tag{44}$$

From (44),

$$\alpha_i = \tfrac{1}{2}R(\omega/\alpha_r - U_1). \tag{45}$$

Let the left-hand side of the inequality (43) be set equal to $-\frac{1}{4}k^2R^2$, with k real and non-zero.

Then, substituting for α_i from (45), it is found that

$$\alpha_r = \{\tfrac{1}{2}[(\beta^2 + \omega^2R^2)^{\frac{1}{2}} - \beta]\}^{\frac{1}{2}} \tag{46}$$

with

$$\beta = \tfrac{1}{4}R^2(U_1^2 + k^2); \tag{47}$$

therefore the continuous spectrum for the spatial stability problem lies on the curve given parametrically by (45)–(47). It can be shown that the positive square root must be taken in (46). If the negative sign is taken, (43) implies that $\alpha_r^2 < 0$, which is impossible. It can further be shown that both $\alpha_r > 0$ and $\alpha_i > 0$. Since $\alpha_i > 0$, all the continuum modes are damped for the spatial stability problem.

It is interesting to examine a few special cases. In most cases of interest $\omega/R \ll 1$.

For a boundary layer, wake or free shear layer, $U_1 = 1$, and it is easily shown from (45)–(47) that, with $\omega/R \ll 1$, as $k \to 0$

$$\alpha_r \to \omega, \quad \alpha_i \to R[\tfrac{1}{4}k^2 + (\omega/R)^2], \tag{48), (49}$$

$$c \approx 1 - i[\tfrac{1}{4}k^2 + (\omega/R)^2]/(\omega/R), \tag{50}$$

while as $k \to \infty$

$$\alpha_r \to \omega/k, \quad \alpha_i \to \tfrac{1}{2}kR, \tag{51), (52}$$

$$c \approx 4(\omega/R)^2 k^{-3} - i2(\omega/R)\,k^{-1}. \tag{53}$$

In all cases c_r, the phase speed, is less than or equal to the free-stream speed. The least-damped modes are those for which $c_r \approx 1$ as $k \to 0$ or $c_r \approx 0$ as $k \to \infty$. The former are progressive waves moving at nearly the free-stream speed and the latter are standing waves.

For a jet $U_1 = 0$ and it is easily seen that, with $\omega/R \ll 1$, as $k \to 0$

$$\alpha \to (\tfrac{1}{2}\omega R)^{\frac{1}{2}}(1+i), \tag{54}$$

$$c \approx (\omega/2R)^{\frac{1}{2}}(1-i), \tag{55}$$

while as $k \to \infty$

$$\alpha \to \omega k^{-1} + \tfrac{1}{2}iRk, \tag{56}$$

$$c \approx 4(\omega/R)^2 k^{-3} - i2(\omega/R)\,k^{-1}. \tag{57}$$

In the limit $k \to \infty$, the same results are obtained for α and c whether $U_1 = 0$ or $U_1 = 1$. The reason for this is that the limiting process is the same, i.e. $U_1/k \to 0$.

3. Application to specific shear flows

3.1. *The Blasius boundary layer*

The flow is that past a semi-infinite flat plate and the velocity profile $U(y)$ is the Blasius boundary-layer profile (Batchelor 1967, p. 308). The velocity and length scales chosen in this case are U_0, the free-stream speed, and

$$L(x) = (\nu x/U_0)^{\frac{1}{2}}, \tag{58}$$

where x is the distance from the leading edge. With this choice of length scale the momentum thickness δ^* and the boundary-layer thickness δ are $1{\cdot}72\,L(x)$ and $5{\cdot}04\,L(x)$, respectively. Since $U(y) \to 1$ as $y \to \infty$, $U_1 = 1$ and for the temporal stability problem the continuous spectrum lies along

$$c_T = 1 - i(1+k^2)\,\alpha/R \tag{59}$$

in the complex wavenumber plane, while for the spatial stability problem the continuous spectrum lies along the curve in the complex wavenumber plane given parametrically by

$$c_s = \omega\alpha^*/|\alpha|^2, \tag{60}$$

$$\alpha_r = \{\tfrac{1}{2}[(\beta^2 + \omega^2 R^2)^{\frac{1}{2}} - \beta]\}^{\frac{1}{2}}, \tag{61}$$

$$\alpha_i = \tfrac{1}{2}R(\omega/\alpha_r - 1) \tag{62}$$

and

$$\beta = \tfrac{1}{4}R^2(1+k^2). \tag{63}$$

Figure 1 shows the complex c plane. The continuous spectra for both the temporal problem ($\alpha = 0{\cdot}179$, $R = 580{\cdot}0$) and the spatial problem ($\alpha = 0{\cdot}04649$, $R = 580{\cdot}0$) are shown as well as the TS eigenvalues computed by Corner *et al.* and Mack (1976).† None of the TS modes found by Mack or Corner *et al.* lie on the continuous spectrum.

We have calculated the continuum eigenfunctions of the temporal problem, for the values of (α, R) given above, at a large number of points on the continuous spectrum. The numerical method is very similar to that used by Mack. For example, consider the temporal problem at a given α and R. A value of k is chosen, and from (59) a point on the spectrum is determined. The upper boundary condition is applied at some large y, say y_0, where (25) is satisfied to at least one part in 10^8. A value of $y_0 = 9$ is sufficient at $\alpha = 0{\cdot}179$, $R = 580$; then at y_0

$$\phi_1 = \exp(-ik\alpha y_0), \quad \phi_2 = \exp(ik\alpha y_0), \quad \phi_3 = \exp(-\alpha y_0). \qquad (64)\text{–}(66)$$

From these equations $\phi_j'(y_0)$, $\phi_j''(y_0)$ and $\phi_j'''(y_0)$ are determined. The Orr–Sommerfeld equation (2) is then numerically integrated to $y = 0$ by a fourth-order Runge–Kutta method. A linear combination of the three independent solutions evaluated at $y = 0$ is then chosen to satisfy the boundary conditions, in this case (19). This leaves one of the three constants undetermined. Choosing this constant effectively determines the normalization. The normalization we have chosen is to set A, the coefficient of ϕ_1, equal to one.

The nature of these eigenfunctions can be seen by considering them in the far field, i.e. as $y \to \infty$. For any k, the continuum stream function is, as $y \to \infty$,

$$\begin{aligned}\psi &\sim \exp[i\alpha(x-ky-ct)] + B\exp[i\alpha(x+ky-ct)] + C\exp(-\alpha y)\exp[i\alpha(x-ct)] \\ &= \exp[-(1+k^2)\alpha^2 t/R]\{\exp[i\alpha(x-ky-t)] + B\exp[i\alpha(x+ky-t)] \\ &\qquad + C\exp(-\alpha y)\exp[i\alpha(x-t)]\}. \qquad (67)\end{aligned}$$

The amplitude of the stream function is decaying exponentially in time with a decay time of $R/[\alpha^2(1+k^2)]$. Apart from this decaying amplitude, the first term in the bracket is a progressive wave, with an amplitude of unity, travelling towards the solid boundary. The magnitude of the wavenumber is $\alpha(1+k^2)^{\frac{1}{2}}$ and the propagation vector lies at an angle $-\theta$ with respect to the x axis, where

$$\theta = \tan^{-1}(k). \qquad (68)$$

The second term is an outgoing wave with complex amplitude B. The wavenumber is of the same magnitude as that of the incoming wave while the propagation vector is at the angle $+\theta$ with respect to the x axis. The third term is a 'wall' wave with complex amplitude C, which decays outwards from the wall. This wave is propagating parallel to the wall with wavenumber α.

In the language of scattering, these continuum modes consist of an incoming wave (amplitude $= 1$, angle of incidence $= \theta$) which interacts with the boundary layer and reflects as an outgoing wave (complex amplitude $= B$, angle of reflexion $= \theta$). A 'wall' wave (complex amplitude $= C$) is generated which, in combination with the incident and reflected waves, satisfies the boundary conditions at $y = 0$. The complex amplitudes B and C are, of course, determined by the detailed shape of the velocity profile

† The values given here differ from those in the paper of Corner *et al.* (1976) because they used the displacement thickness as the length scale, instead of L as given in (58). Mack used L as the length scale.

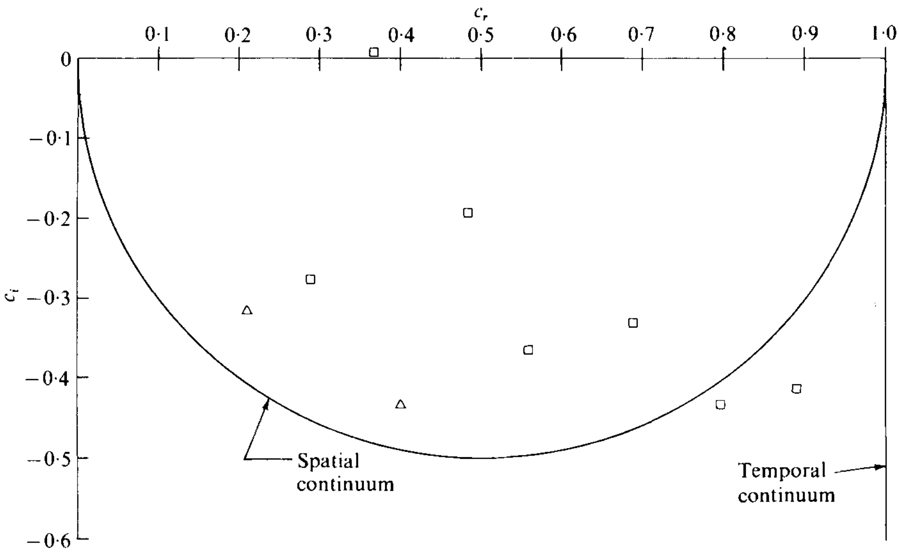

FIGURE 1. The spatial and temporal spectrum of the Orr–Sommerfeld equation for Blasius flow at $R = 580{\cdot}0$ and $\omega = 0{\cdot}04649$ (spatial) or $\alpha = 0{\cdot}179$ (temporal). Points of the discrete spectrum at this R and ω or α: $\triangle$, as computed by Corner *et al.* (1976) (spatial modes); $\square$, results of Mack's (1976) calculations (temporal modes).

through the Orr–Sommerfeld equations and the boundary conditions at $y = 0$. For a fixed α and R, B and C also vary with k.

The magnitudes and phases of the complex amplitudes B and C are shown in figure 2 as a function of θ, the angle of incidence of the incoming wave, for the temporal stability continuum at $\alpha = 0{\cdot}179$, $R = 580{\cdot}0$. The magnitudes of B and C are plotted in figure 2 (a) and the phases in figure 2 (b).

As $\theta \to 0$, i.e. at grazing incidence, $|B| \to 1$, $\mathrm{ph}\,B \to \pi$, $|C| \to 0$ and $\mathrm{ph}\,C \to -99°$. As θ increases, $|B|$ and $\mathrm{ph}\,B$ both decrease slowly up to $\theta \approx 65°$. In the same range $|C|$ increases and $\mathrm{ph}\,C$ decreases. The maximum of $|C|$ occurs at $\theta \approx 65°$, where $|C| \approx 3{\cdot}7$ and $\mathrm{ph}\,C \approx -\frac{3}{2}\pi$. For larger θ, $|C|$ decreases very rapidly, and both $\mathrm{ph}\,B$ and $\mathrm{ph}\,C$ decrease rapidly. The variations of $\mathrm{ph}\,B$ and $\mathrm{ph}\,C$ are shown only up to $\theta \approx 85°$. The decrease in the phase with θ is so rapid for larger angles of incidence that it is not possible to show it on this figure.

The real and imaginary parts of ϕ are shown in figure 3 as a function of y for the case $\alpha = 0{\cdot}179$, $R = 580{\cdot}0$ and $k = 2{\cdot}0$ ($\theta = 63{\cdot}43°$). It can be seen that the disturbance is very small within the lower part of the boundary layer (the top of the boundary layer is at $y = 5{\cdot}04$). This was found to be true for all θ. These continuum modes are essentially free-stream modes and do not penetrate very far into the boundary layer.

Contours of the stream function ψ in the far field for the same case are shown in figure 4. The region shown is $0 \leqslant \alpha x \leqslant 4\pi$, $10\pi < \alpha y < 14\pi$. It can be seen that these continuum modes are a doubly periodic array of vortices in the free stream. The ratio of the wavelength in the x direction to that in the y direction is k, in this case $2{\cdot}0$. Because $|A| \neq |B|$, the axes are tilted and the vortices are distorted. The vortex array is moving with the free stream, i.e. has a phase speed of $1{\cdot}0$, and the vortex strength is decaying in time as $\exp[-(1+k^2)\alpha^2 t/R]$. It is readily seen that all the

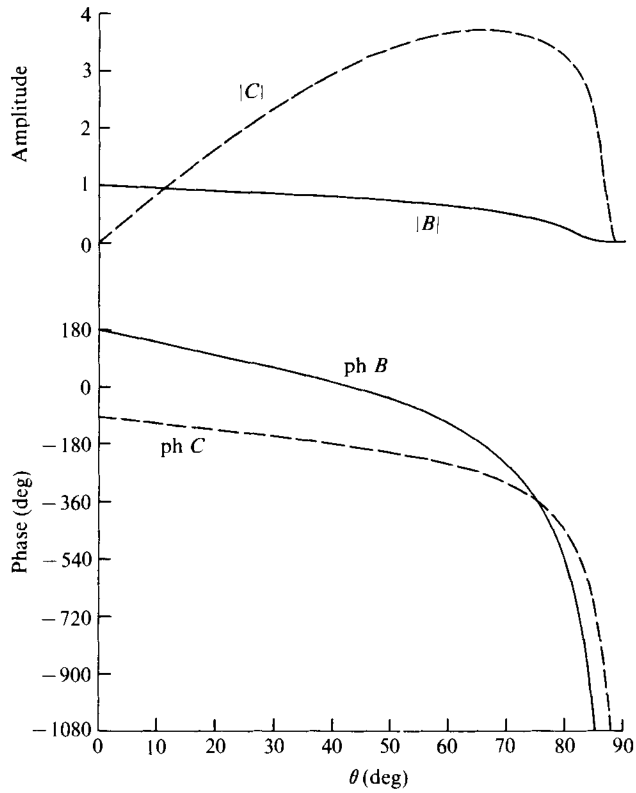

FIGURE 2. The magnitudes and phases of the complex amplitudes B and C for the temporal continuum modes of the Blasius boundary layer at $\alpha = 0{\cdot}179$, $R = 580$.

continuum modes are of the same general form in the far stream. The aspect ratio of the vortices and the distortion depend on k and $B(k)$. If k and B were both equal to 1·0, this flow field would be the well-known Taylor (1923) vortex solution.

Some related results have been obtained by Rogler (1975) and Rogler & Reshotko (1975), who considered the response of a boundary layer to an array of non-decaying free-stream vortices. This problem is not, as emphasized by Rogler & Reshotko, an eigenvalue problem, because the form of the vortex array is imposed in the far field, as well as the requirement that the vorticity be constant in time. Applying a perturbation analysis to the Navier–Stokes equations, Rogler & Reshotko obtained, formally, an inhomogeneous Orr–Sommerfeld equation with a forcing function depending on the mean velocity field and the vortex array. It should be emphasized that the vortex arrays considered by Rogler & Reshotko are not the continuum eigenfunctions of the Orr–Sommerfeld equation. The connexion between these continuum modes and the free-stream vortices used by Rogler & Reshotko will be discussed in a later paper (part 2).

3.2. *The two-dimensional (plane) laminar jet*

As a second example, the continuum modes of the two-dimensional laminar jet will be discussed. The velocity of this jet is, in dimensionless units,

$$U(y) = 1 - \tanh^2 y. \tag{69}$$

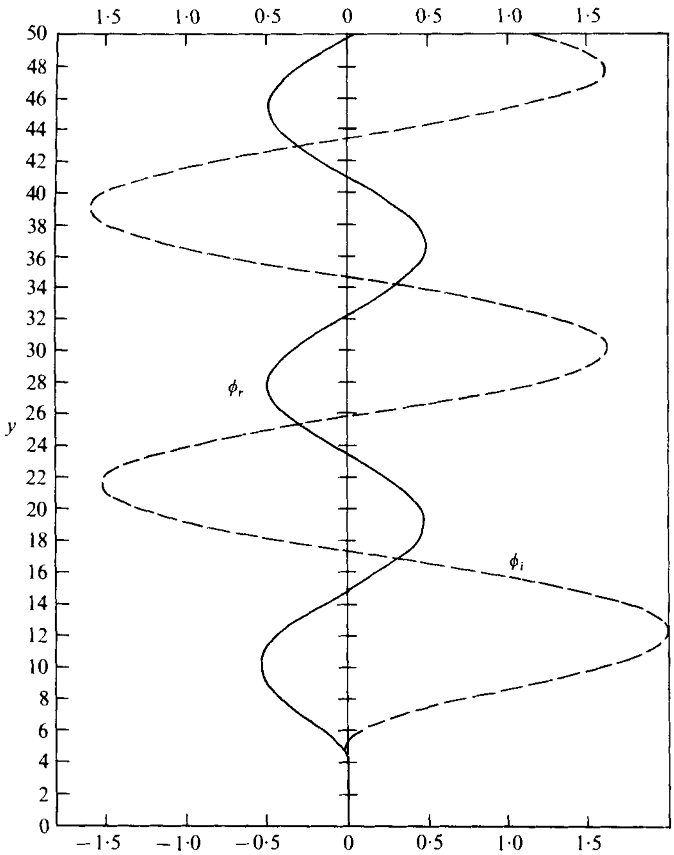

Figure 3. The real part $\phi_r(y)$ (solid line) and imaginary part $\phi_i(y)$ (dashed line) of $\phi(y)$ *vs.* y for the temporal continuum mode of the Blasius boundary layer at $\alpha = 0{\cdot}179$, $R = 580$ and $k = 2$ ($\theta = 63{\cdot}43°$).

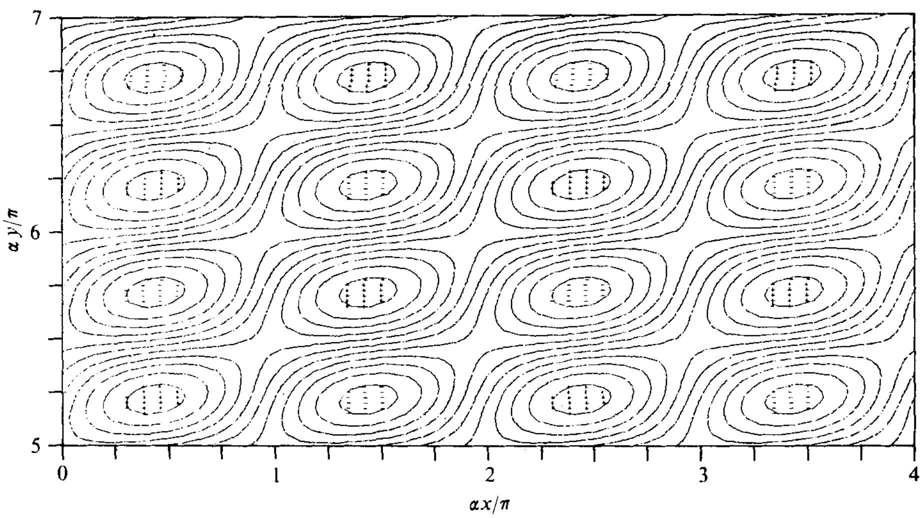

Figure 4. The far-field stream function of the temporal continuum mode of the Blasius boundary layer at $\alpha = 0{\cdot}179$, $R = 580$ and $k = 2$ ($\theta = 63{\cdot}43°$).

The length scale $L(x)$ and velocity scale $U_0(x)$ are

$$L(x) = [48\nu^2 x^2/K]^{\frac{1}{3}} \tag{70}$$

and

$$U_0(x) = [3K^2/32\nu x]^{\frac{1}{3}}, \tag{71}$$

where

$$K = \int_{-\infty}^{\infty} \hat{U}(\hat{y})\, d\hat{y} \tag{72}$$

is the momentum flux per unit density per unit width of the jet (Schlichting 1951, p. 164), the carets denoting dimensional quantities.

It is clear that the parallel shear flow approximation is poorer for a jet ($L(x) \sim x^{\frac{2}{3}}$) than for a boundary layer ($L(x) \sim x^{\frac{1}{2}}$), but we shall not consider here the modification to the continuum necessary in the non-parallel flow approximation (but see § 4).

The free-stream speed is zero for the jet, and so the continuous spectrum for the temporal stability problem lies along the line

$$c_T = -i(1+k^2)\,\alpha/R. \tag{73}$$

As discussed above, this symmetric jet has two continuum modes, the symmetric and antisymmetric modes, for each point on the continuum. In both cases the modes are decaying standing waves.

The continuum of the spatial stability problem for the jet lies along the curve

$$\alpha_r = (\tfrac{1}{2}kR)\{\tfrac{1}{2}[1+16(\omega/k^2R)^2)^{\frac{1}{2}}-1]\}^{\frac{1}{2}}, \tag{74}$$

$$\alpha_i = \omega k^{-1}\{\tfrac{1}{2}[(1+16(\omega/k^2R)^2)^{\frac{1}{2}}-1]\}^{-\frac{1}{2}}. \tag{75}$$

In contrast to the temporal stability problem, the continuum modes of the jet for the spatial stability problem are spatially decaying travelling waves with a wave speed

$$c = \omega/\alpha_r = (2\omega/kR)\{\tfrac{1}{2}[(1+16(\omega/k^2R)^2)^{\frac{1}{2}}-1]\}^{-\frac{1}{2}}. \tag{76}$$

We have calculated the symmetric and antisymmetric continuum modes of the laminar jet for the temporal stability problem at $\alpha = 1{\cdot}0$, $R = 50{\cdot}0$ and $0 \leqslant k \leqslant 100{\cdot}0$ ($0 < \theta \lesssim 89{\cdot}4°$). The numerical method is identical to that used for the boundary layer.

The amplitudes and phases of B and C for the antisymmetric modes are plotted *vs.* θ in figure 5 for the case $\alpha = 1{\cdot}0$ and $R = 50$. We have also calculated the corresponding amplitudes and phases for the symmetric modes at this value of (α, R). Although the forms of the eigenfunctions are different near $y = 0$ for the symmetric and antisymmetric modes, the differences between the numerical results for the amplitudes and phases are so small that they cannot be shown on this figure.

From this figure it can be seen that, as $\theta \rightarrow 0$, $|B| \rightarrow 1{\cdot}0$ and $|C| \rightarrow 0$, while ph$B \rightarrow 180°$ and ph$C \rightarrow -45°$. The amplitudes of both B and C rise monotonically with increasing θ to a peak response at $\theta \approx 79{\cdot}5°$ ($k \approx 5{\cdot}15$) and then decrease rapidly. The phases of B and C decrease monotonically with increasing θ, slowly for $\theta \lesssim 79{\cdot}5°$ and then very rapidly. The phases are not shown for $\theta > 85°$ because the extremely rapid variation with θ in this region makes it impractical.

The maximum amplitudes of B and C are very large for the jet as compared with the boundary layer when it is noted that in both cases the normalization is the same, i.e. $A = 1$ for all θ. In the case of the jet, the maximum amplitude of B is about 4850 (1·0 for a boundary layer) and the maximum amplitude of C is about 2300 (3·7 for a

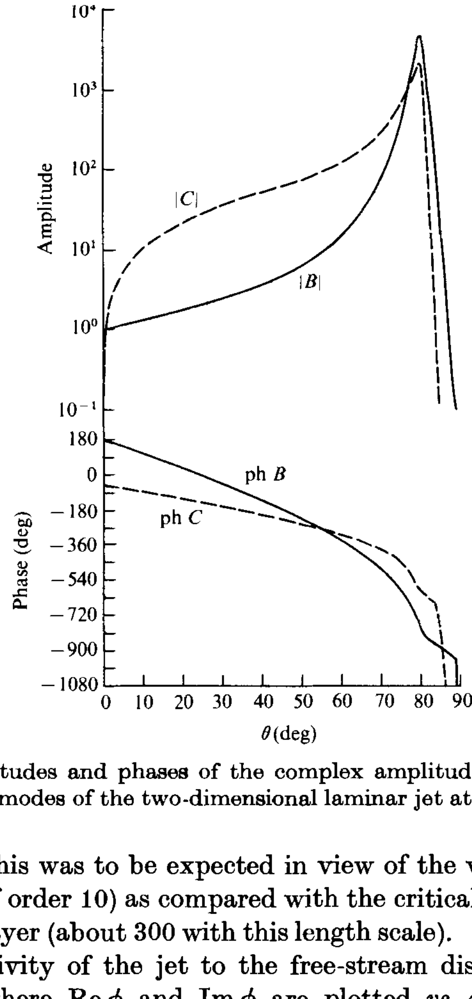

FIGURE 5. The magnitudes and phases of the complex amplitudes B and C for the temporal continuum modes of the two-dimensional laminar jet at $\alpha = 1$ and $R = 50$.

boundary layer). This was to be expected in view of the very low critical Reynolds number for a jet (of order 10) as compared with the critical Reynolds number for the Blasius boundary layer (about 300 with this length scale).

This same sensitivity of the jet to the free-stream disturbances can be seen in figures 6 and 7, where $\mathrm{Re}\,\phi$ and $\mathrm{Im}\,\phi$ are plotted *vs.* y for the symmetric and antisymmetric modes with $\alpha = 1{\cdot}0$, $R = 50{\cdot}0$ and $k = 2{\cdot}0$ ($\theta = 63{\cdot}43°$). It can be seen that in both cases the disturbance penetrates much further into the jet than was the case for the boundary layer. If the 'edge' of the undisturbed jet is defined as the point where $U(y) \approx 0{\cdot}001$, then the jet occupies the region $|y| \lesssim 3$, and it is seen from figures 6 and 7 that there is quite a substantial disturbance in this region. If the amplitude of the disturbance were large enough the apparent 'edge' of the jet could be moved by the disturbance.

Figure 8 shows the far-field stream function for the symmetric mode of the jet at $\alpha = 1{\cdot}0$, $R = 50{\cdot}0$ and $k = 2{\cdot}0$. The region shown is $0 \leqslant \alpha x \leqslant 8\pi$, $10\pi \leqslant \alpha y \leqslant 14\pi$. As in the case of the Blasius boundary layer, the far field is a doubly periodic array of vortices, but in contrast to the boundary layer the vortices are highly distorted since $|B|$ is considerably different from $|A| = 1$. At this distance ($y > 30$) from the jet axis the trapped mode is negligible. The far-field stream function for the antisymmetric mode is essentially identical to that for the symmetric mode and is not shown.

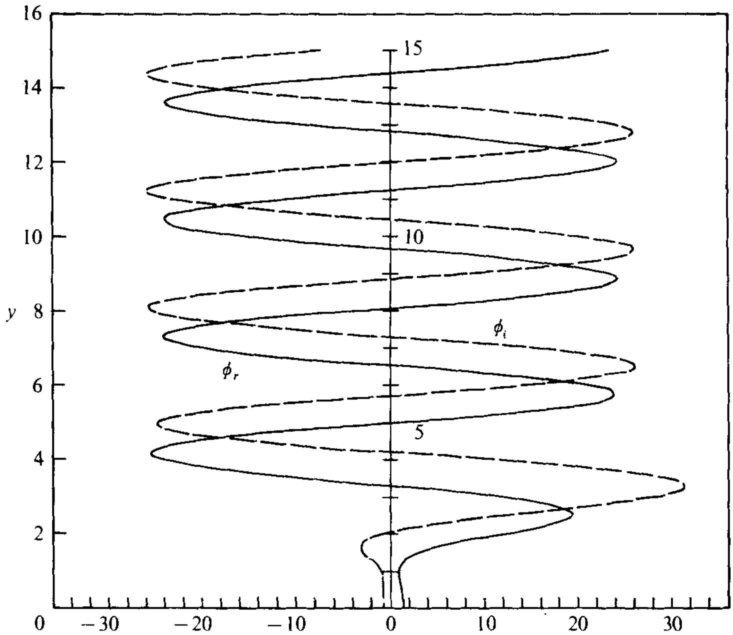

FIGURE 6. The real part $\phi_r(y)$ (solid line) and imaginary part $\phi_i(y)$ (dashed line) of $\phi(y)$ for the symmetric temporal stability mode of a two-dimensional laminar jet at $\alpha = 1$, $R = 50$ and $k = 2$ ($\theta = 63{\cdot}43°$).

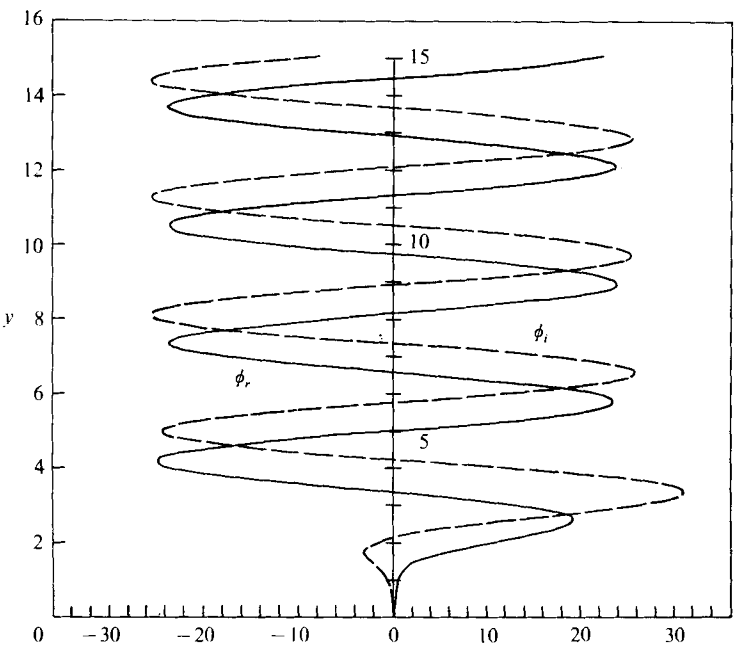

FIGURE 7. The real part $\phi_r(y)$ (solid lino) and imaginary part $\phi_i(y)$ (dashed line) of $\phi(y)$ for the antisymmetric temporal stability mode of a two-dimensional laminar jet at $\alpha = 1$, $R = 50$ and $k = 2$ ($\theta = 63{\cdot}43°$).

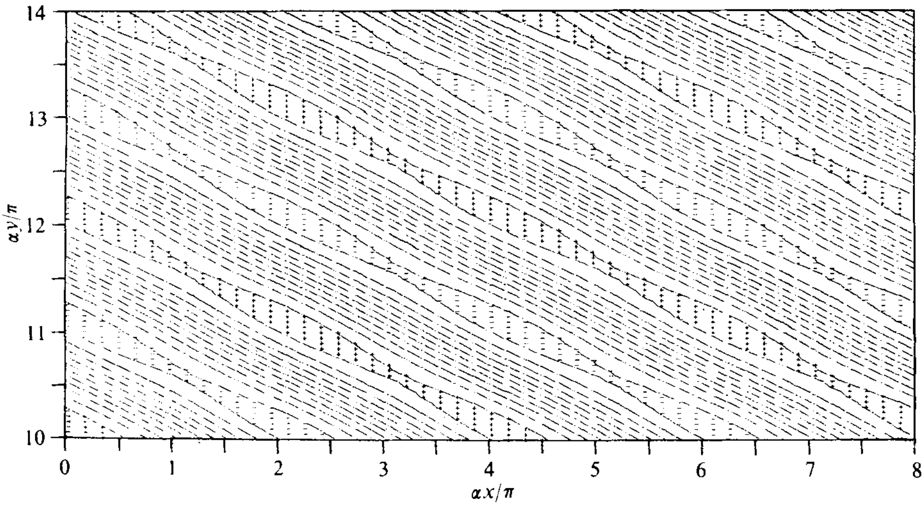

FIGURE 8. The far-field stream function of the symmetric temporal continuum mode of the laminar jet at $\alpha = 1$, $R = 50$ and $k = 2$ ($\theta = 63{\cdot}43°$).

4. The continuum modes and transition

All the continuum modes, both temporal and spatial, for the linearized stability problem of an unbounded parallel shear flow are damped; the flow is linearly stable with respect to these modes. It is, of course, possible that if the free-stream vorticity were sufficiently intense the nonlinear problem would yield growing solutions, i.e. the flow would be unstable to these modes and transition would result.

We believe, however, that there are three possible mechanisms by which *small amplitude* continuum modes might lead to transition. The first mechanism involves a small quasi-steady distortion of the mean profile by the continuum modes, the second mechanism requires the consideration of the fully time-dependent flow, consisting of the mean flow plus the slowly decaying continuum modes, while the third mechanism involves consideration of the non-parallel flow corrections to the continuum spectrum.

The usual picture of transition via TS waves in a boundary layer consists of: the appearance of a small amplitude, unstable TS wave of unspecified origin; its growth through the region of validity of the linear theory; the appearance of small nonlinear effects, i.e. higher harmonics and, more important, a small three-dimensional distortion of the mean profile near the top of the boundary layer; and finally the appearance of short wavelength, rapidly growing TS waves which lead directly to the appearance of a turbulent burst. It is believed that the penultimate stage, the distortion of the mean profile near the top of the boundary layer, is crucial. This distortion, in the region where U'' is small, causes U'' to change sign, and the inflexion in the mean profile permits the appearance of small wavelength, highly unstable TS waves. These, presumably, become nonlinear rather quickly and cause a further distortion of the mean profile, and so on to transition. This is, in effect, a modified Landau model.

Now consider the effect of a small patch of vorticity in the free stream. Assuming, as we shall show in part 2, that we can represent this by an integral over the continuum, continuum modes of all wavelengths are present and exciting the boundary layer.

Among these are small α, small k continuum modes, i.e. lightly damped, long wavelength and hence low frequency modes. In a quasi-steady model these produce a small quasi-steady distortion of the mean profile in the outer regions where U'' is small; therefore we might expect that the boundary layer has reached the critical stage when short wavelength, highly unstable TS waves can appear without a long wavelength, slowly growing TS wave ever being experienced. In effect, this mechanism would bypass the linear TS stage and appear to lead directly to the late stages of transition.

The second mechanism involves consideration of the fully time-dependent flow, consisting of a superposition of the mean flow and the small amplitude continuum modes. Again consider the effect of a small patch of free-stream vorticity. Among the modes necessary to represent this patch there will be small α modes, i.e. long wavelength, lightly damped waves. Let us consider the stability of this time-dependent flow: the mean flow plus the lightly damped modes. Relatively little is known about the stability of time-dependent shear flows (see Davis (1976) for a recent review), but it is known that under certain circumstances these flows can have resonances. That is, if the 'matching' is proper, the time-dependent flow can be unstable to TS-like modes which extract energy from the mean flow at a rapid rate; they are highly unstable. The unstable mode tends to lie near a higher mode of the steady Orr–Sommerfeld equation (Grosch & Salwen 1968). Again, if this were to occur, the linear TS region of instability would be bypassed and it would appear, in an experiment, that the short wavelength, highly unstable waves had been directly excited.

Finally, let us consider the possible effects of non-parallel flow corrections on these continuum modes. Saric & Nayfeh (1975) have developed a general theory which permits the calculation of the correction to the spectrum of the Orr–Sommerfeld equation due to the non-parallel nature of any nearly parallel shear flow. They have applied this theory to the stability of Falkner–Skan flows. In all cases which they considered, the non-parallel flow corrections resulted in a decrease in the stability of the boundary layer compared with the parallel flow calculation at fixed (real) wavenumber α and Reynolds number R. That is, at a fixed α, the flow was predicted to be unstable at a lower R than that predicted by the parallel shear flow approximation; also, at a fixed R, the band of unstable wavenumbers was wider than that predicted from the usual parallel flow approximation.

The continuum modes, particularly at the long wavelength end of the spectrum, are very lightly damped. It does not appear to be unreasonable that non-parallel flow corrections may make a portion of the continuous spectrum unstable.

All of these mechanisms are, at this time, only hypotheses. We know of no direct evidence for or against any of them, other than the observations that free-stream turbulence can trigger transition without the appearance of the usual TS waves. We believe, however, that all of these mechanisms are sufficiently plausible that experimental and theoretical tests of them are warranted. We hope to carry out a theoretical investigation of these proposed transition mechanisms; we are currently investigating the non-parallel flow corrections.

5. Summary

We have shown that the Orr–Sommerfeld equation (for both the temporal and the spatial problem) has a continuous spectrum for any mean flow which is an unbounded shear flow and has finite energy under some Galilean transformation.

Formulae for the location of the continuum in the complex wave-speed plane have been given. These results have been applied to two specific flows: the Blasius boundary layer and the laminar jet. For both of these flows the continuous spectra have been given. The continuum eigenfunctions of the temporal problems have been calculated for both flows for a single wavenumber and Reynolds number. It has been shown that the eigenfunctions in the free stream are a doubly periodic array of vortices and that these eigenfunctions can be thought of as an incident wave, a reflected wave and a 'wall' wave. If the amplitude of the incident wave is held fixed, the amplitudes and phases of the reflected and 'wall' waves vary with the angle of incidence, and there is an optimum angle of incidence which maximizes the amplitude of the wall wave. It was shown that the continuum modes do not penetrate deeply into the region of large shear in the main flow of a boundary layer although they penetrate further and have much larger amplitudes in the jet. Finally, three mechanisms were proposed by which these continuum modes could cause transition in a shear flow while bypassing the usual linear TS stage. These mechanisms are at present hypotheses; we hope to investigate these hypotheses theoretically as well as to investigate the relation of these eigenfunctions to the forced free-stream disturbances studied by others. We hope that experimentalists will also look for these modes.†

We wish to thank both referees for their helpful comments. In particular, we wish to thank Dr Leslie M. Mack for forcing us to use double precision in the calculations for figure 2, thereby removing some serious errors. This research was supported by the Advanced Research Projects Agency of the Department of Defense and was monitored by ONR under Contract no. N00014-75-C-0777 to the first author.

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† *Note added in proof.* We have recently found that the spatial continuum discussed here is only one branch of a four-branched continuum. The existence of four branches is due to the fact that the linearized equation for the stream function is of fourth order in x (Grosch & Salwen 1978).

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